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TWO NONISOMORPHIC K-AUTOMORPHISMS WITH ISOMORPHIC SQUARES[†]

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ABSTRACT

By taking two different skew products of an initial transformation and a two point space, two measure preserving transformations with the same square are constructed. By direct arguments on the doubly infinite partition names of points in these processes, they are shown to be K-automorphisms and nonisomorphic.

1. Introduction

Through Ornstein's work on the isomorphism theory of Bernoulli-shifts [3], [4], [5], it is known that any square root of a Bernoulli-shift is Bernoulli. Hence any two such roots are isomorphic. Ornstein has also constructed a K-automorphism which has no square roots [7], and J. Clark has extended this to a K-automorphism with no roots at all [2]. Our goal here is to construct two K-automorphisms which are nonisomorphic but have isomorphic squares. The square, thus, has two nonisomorphic square roots. These transformations are given as two skew products of a K-automorphism that is not Bernoulli with a two point space. We use the arguments developed by Ornstein and Shields [6] in their construction of uncountably many nonisomorphic K-automorphisms to show they are K-automorphisms and are nonisomorphic.

The thought behind this construction is the following trivial example of two transformations with isomorphic squares. Let Ω' be a two point space, T_1 the interchange map and T_2 the identity map. Obviously $T_1^2 = T_2^2$, as both are the identity map. The maps which we construct will be skew products with a K-automorphism which preserve this identity on the square.

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2. Construction of T_1 and T_2

We will construct first an initial transformation W following the method used by Ornstein and Shields [6]. Hence, we will be sketchy in some of the details of the construction and refer the reader to this paper or Ornstein's text, *Ergodic Theory, Randomness and Dynamical Systems* (see [8]) for a more careful consideration of such constructions.

The map W will be defined as the shift transformation on doubly-infinite names from a partition (E, F, S_0, S). The names are built up through a *block structure.* We will describe how to make the initial 0-*blocks*, and state inductively how $(n-1)$-block names are strung together to form n-block names.

A 0-block name will consist of two F's followed by 2^{100} S_0's, followed by two E's. An n-block name will be made of $(n-1)$-block names as follows. Select independently a sequence of 2^{2n} $(n-1)$-block names. Pick a value $f \in \{1 \cdots n+1\}$ independent of the $(n-1)$-block names chosen, each with equal probability. For this value of f and this sequence of $(n-1)$-block names, Fig. 1 shows what the n-block name will look like.

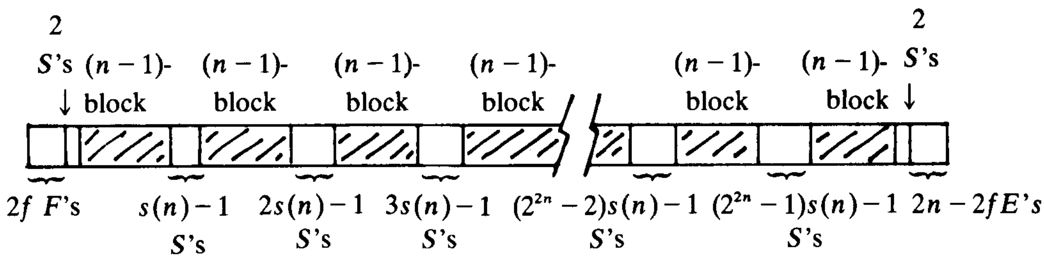

Fig. 1

This gives the various possible n-block names. Each is equally likely, and the number of F's at the beginning of the block and which $(n-1)$-blocks occur in the name are equidistributed over their possible values and independent of each other. If we let $f(n) = 2(n+1)$, the variability of the F segment, $s(n) = 100n^3$, the increment size in the spacers, and $h(n)$ be the length of an n-block, then we have

$$\sum_{i=0}^{n} f(i) < s(n) - 1 \quad \text{and} \quad 2^{10n}(s(n)-1) < h(n-1).$$

Further, if we let Ω_n be the set of all points in an n-block, then $\mu(\Omega_0) > \frac{1}{2}\mu(\Omega)$, where Ω is the entire space on which W is defined.

To construct T_1 take the direct product of Ω with the two point space, $\{0, 1\}$.

Call this space $\bar{\Omega}$. On $\bar{\Omega}$ let P be the partition $(E \times \{0,1\},\ F \times \{0,1\},\ S \times \{0,1\},\ S_0 \times 0,\ S_0 \times 1) = (\bar{E}, \bar{F}, \bar{S}, r, b)$. We will often speak of a point in r or b as colored red or black. For a point of this product T_1 will be defined by stating how the color of a point changes as it moves along an n-block. That is to say, $T_1(w, *) = (W(w), S_w(*))$, where S_w is either the interchange or the identity map, and either fixes or switches the color. There are two colorings for 0-blocks given in Fig. 2.

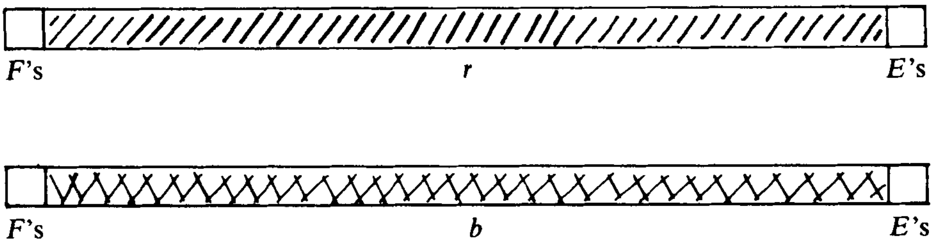

Fig. 2

Suppose we have shown how to color the red and black $(n-1)$-blocks. Fig. 3 now shows how to color a red and a black n-block. By a red n-block we mean an n-block whose first 0-block is red.

This prescription tells us whether, at the end of an $(n-1)$-block in an n-block, S_w is the switch or the identity map. Thus a red n-block is a sequence of $(n-1)$-blocks, switching between red and black $(n-1)$-blocks in the order rbbrrbbr $\cdots$ rbbr, and a black n-block is colored in just the opposite order, brrbbrrb $\cdots$ brrb. This defines T_1.

We can define the map T_2 as follows. Let T' be the interchange map on the second coordinate of $\Omega \times \{0,1\}$. Now let $T_2(\bar{w}) = T_1(T'(\bar{w}))$ for all $\bar{w} \in \bar{\Omega}$. As

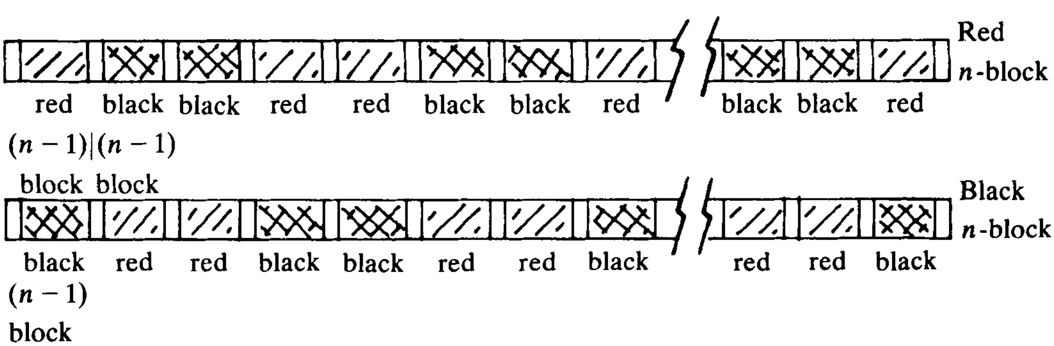

Fig. 3

$T_1T' = T'T_1$, we get $T_2^n = T_1^nT'^n$. Looking at Fig. 3, whenever T_1 switches the $\{0, 1\}$ coordinate of $\bar{w}$, T_2 fixes it and vice versa. Hence to get a colored n-block T_2 name, we take an n-block T_1 name and interchange the colors in odd positions and fix those in even.

Thus there are two possible colored 0-block T_2 names, as in Fig. 4.

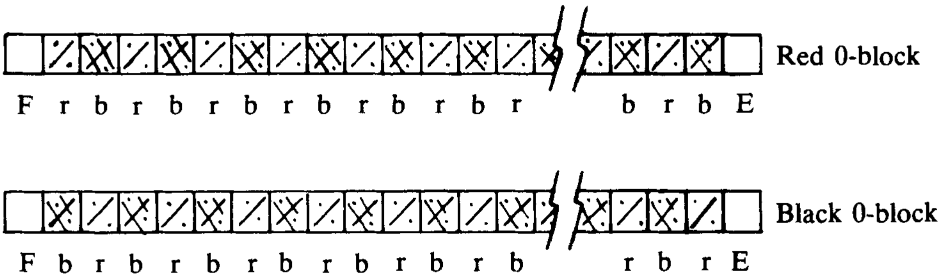

Fig. 4

Red and black T_2 $(n-1)$-blocks string together to form a red and a black T_2 n-block, but notice that as the F's always come in even blocks, and $ks(n)$-1 is odd, that the possible color sequences are, for a red T_2 n-block, rrbbrrbbrr $\cdots$ bb, and for a black T_2 n-block, bbrrbbrrbb $\cdots$ rr. It is this difference between T_1 and T_2 names which will force T_1 and T_2 to be nonisomorphic. The following fact is now trivial.

THEOREM 1.1. *T_1^2 and T_2^2 are isomorphic.*

PROOF. We have even more, that $T_2^2 = T_1^2T'^2 = T_1^2$.

3. T_1 and T_2 are K-automorphisms

As T_1^2 and T_2^2 are the same, and roots and powers of K-automorphisms are K-automorphisms, it will suffice to show T_1 is a K-automorphism.

Our argument will follow the format of the similar result in Ornstein and Shields [6], only with a minor variation to cope with the evenness of the F-sections and the different colorings. Let P be the partition $\{\bar{E}, \bar{F}, \bar{S}, r, b\}$. We will, as is usual, prove the following version of the K property.

THEOREM 1.2. *Given any $\varepsilon > 0$ and integer k, there is an N such that for any $m > 0$ and $n > N$,*

$$\bigvee_{-m}^{0} T_1^i(P) \perp^{\varepsilon} \bigvee_{n-k}^{n+k} T_1^i(P).$$

PROOF. First fix $\varepsilon > 0$ and the integer k. Choose N_0 so large that the set of all w in an N_0-block, more that k positions from either end, is at least $(1 - \varepsilon^2)$. Call this subset $\tilde{\Omega}$. Let L be so large that $f(L) > h(N_0 + 1)$, and let $N = h(L) + 1$.

Now fix $n > N$ and let $\tilde{\Omega}^n = T^{-n}(\tilde{\Omega})$. As $\mu(\tilde{\Omega}) > 1 - \varepsilon^2$, it will suffice to show

$$\bigvee_{-m}^{0} T_1^i(P)/\tilde{\Omega}^n \perp \bigvee_{n-k}^{n+k} T_1^i(P)/\tilde{\Omega}^n.$$

Let A be an atom of $\bigvee_{-m}^{0} T_1^i(P)$, i.e. a set with a fixed P-name from $-m$ to 0. We want to show that the distribution of T, P-names from $T_1^{n-k}(A)$ to $T_1^{n+k}(A)$ is the same as that on $T_1^{n-k}(\tilde{\Omega}^n)$ to $T_1^{n+k}(\tilde{\Omega}^n)$ i.e. on $T_1^{-k}(\tilde{\Omega})$ to $T_1^{k}(\tilde{\Omega})$.

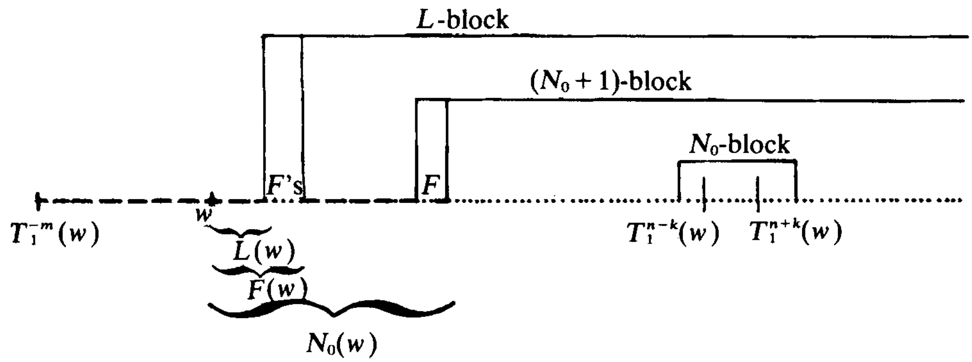

Fig. 5

Define the functions $L(w)$, $F(w)$ and $N_0(w)$ as follows. $L(w)$ is the largest positive integer less than n such that $T_1^{L(w)}(w)$ is the first point of an L-block. $F(w)$ is the first integer larger than $L(w)$ such that $T_1^{F(w)}(w) \notin F$. $N_0(w)$ is the largest integer less than n not in an N_0-block and in F. These are all functions of the name only and hence are measurable. Notice that the size of n makes all of these well defined on $\tilde{\Omega}$. Fig. 5 indicates the situation diagramatically.

Partition A into sets A_i which have a fixed P-name from $T_1^{-m}(w) \cdots T_1^{L(w)}(w)$, and from $T_1^{F(w)}(w) \cdots T_1^{N_0(w)}(w)$ (these sections are dashed in the diagram).

Thus, in A_i, $T_1^n(w)$ must lie in an L-block with a fixed P-name from $F(w)$ to $N_0(w)$, hence always in an $(N_0 + 1)$-block in the same position in the L-block and with a fixed number of F's at its beginning. The F section at the beginning of the L-block, though, can take on any even value, subject only to the restriction that this value will allow $T_1^{n-k}(w) \cdots T_1^{n+k}(w)$ to be in an N_0-block in the proper $(N_0 + 1)$-block. The length of this F section is independent of the N_0-block names in this $(N_0 + 1)$-block, and as $f(L) > h(N_0 + 1)$, will put $T_1^n(w)$ in all the

N_0-blocks, at all allowed positions with equal probability, that is, always at even positions, or odd positions.

As half the N_0-blocks are at an odd distance into the (N_0+1)-block and half even, the independence of the uncolored N_0-block names from the size of this F section now implies that $\bigvee_{n-k}^{n+k} T_1^i(\bar{E}, \bar{F}, \bar{S}_0, \bar{S})/A_i$ has the same distribution of names as $\bigvee_{n-k}^{n+k} T_1^i(\bar{E}, \bar{F}, \bar{S}_0, \bar{S})/\tilde{\Omega}^n$. This will imply that T_1 on uncolored names is K. What about the colored names? Notice that exactly half the odd position N_0-blocks and half the even position N_0-blocks are red. As the uncolored P-name of $T_1^{n-k}(w)\cdots T_1^{n+k}(w)$ will lie in any even position N_0-block with equal probability, it will be in a red or black even block with equal probability. The same holds for odd blocks. Hence a name in $\bigvee_{n-k}^{n+k} T_1^i(\bar{E}, \bar{F}, \bar{S}_0, \bar{S})/A_i$ is in a black N_0-block half the time, and in a red N_0-block half the time. Thus $\bigvee_{n-k}^{n+k} T_1^i(P)/A_i$ has the same distribution of names as $\bigvee_{n-k}^{n+k} T_1^i(P)/\tilde{\Omega}^n$. The same, then, holds for A, the union of the A_i, and the result follows.

4. T_1 and T_2 are nonisomorphic

The thrust of this argument is to show that any isomorphism between T_1 and T_2 must preserve so much of the block structure of their P-names that the fact that the color orders of $(n-1)$-blocks in n-blocks are different in the two will give a contradiction. We begin by showing that in a T_1, P-name, the only thing that looks like an n-block is an n-block. Notice that many of the arguments here are only slightly affected by the colorings.

Let x and y be any two points in Ω. Each has a T_1 (or T_2), P-name which we can write as $\{A_i\}_{i=-\infty}^{\infty}$ and $\{B_i\}_{i=-\infty}^{\infty}$. In these names will occur n-blocks, i.e. sequences $A_k \cdots A_{k+h(n)-1}$ and $B_{k'} \cdots B_{k'+h(n)-1}$ which are T_1 or T_2, P-names across n-blocks. The following facts are in terms of such indexed names.

LEMMA 1.1. *Let $A_k \cdots A_{k+h(n)-1}$ and $B_{k'} \cdots B_{k'+h(n)-1}$ be two $T_1(T_2)$, P-names across n-blocks $|k-k'| = h(n) - L$. If L is such that $h(n) - \Sigma_{i=0}^{n} f(i) > L > h(n)/2^{n+1}$, then $A_i \neq B_i$ for at least $\bar{\varepsilon}L$ values $i \in \{\sup(k,k') \cdots h(n) - \inf(k,k')\}$, where $\bar{\varepsilon}$ is independent of n.*

PROOF. Let

$$\varepsilon_n = \inf_{\text{such } L} \; \inf_{\substack{n\text{-block names} \\ A \text{ and } B}} \left(\frac{\text{number of places } A_i \neq B_i}{L} \right).$$

It is clear $\varepsilon_0 \geqq 1/h(0)$.

Now consider two n-blocks A and B overlapping in L places. As $2^{2n}s(n) < h(n-1)/2^{8n}$, any $(n-1)$-block in A and in the overlap, either overlaps a single $(n-1)$-block in B in at least

$$\left(1-\frac{1}{2^n}\right)h(n-1) - 2^{3n}s(n) \geqq \left(1-\frac{1}{2^n}-\frac{1}{2^{8n}}\right)h(n-1)$$

places, or it overlaps two consecutive $(n-1)$-blocks, each in at least $h(n-1)/2^n$ places, but a total overlap of at least $(1-(1/2^{8n}))h(n-1)$ places. If any $(n-1)$-block in A overlaps an $(n-1)$-block in B in more than $h(n-1)-\Sigma_{i=0}^{n-1}f(i)$ places, then it is the unique such one as $s(n) > \Sigma_{i=0}^{n}f(n)$, $2^{2n}s(n) < h(n-1)$, and $L < h(n) - \Sigma_{i=0}^{n}f(i)$. Hence, except for possibly this one block, and the one block at the end of the overlap, all $(n-1)$-blocks in the overlap satisfy the condition for the definition of ε_{n-1}. Thus the number of $A_i \neq B_{i-(h(n)-L)}$ is

$$\geqq \Bigg(\underbrace{L\left(\frac{\mu(\Omega^{n-1})}{\mu(\Omega_n)}\right)}_{\substack{\text{number of}\\ i \text{ in } (n-1)\text{-}\\ \text{blocks in}\\ \text{overlap}}} - \underbrace{2h(n-1)}_{\substack{\text{blocks}\\ \text{with bad}\\ \text{overlap}}}\Bigg)\varepsilon_{n-1}\underbrace{\left(1-\frac{1}{2^n}-\frac{1}{2^{8n}}\right)}_{\substack{\text{size of}\\ \text{overlap}}}$$

$$> \varepsilon_{n-1}L\left(\frac{\mu(\Omega_{n-1})}{\mu(\Omega_n)}\right)\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{2^n}-\frac{1}{2^{8n}}\right),$$

as

$$\frac{\mu(\Omega_{n-1})}{(\Omega_n)} > \frac{1}{2} \quad \text{and} \quad \frac{h(n-1)}{L} < \frac{1}{2^{2n}}.$$

Taking the infimum over all such L, A and B, and recalling $\mu(\Omega_0)/\mu(\Omega_n) > \frac{1}{2}$

$$\varepsilon_n > \varepsilon_0 \tfrac{1}{2}\prod_{i=1}^{n}\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{2^n}-\frac{1}{2^{8n}}\right).$$

This last sequence is bounded away from zero and we have the result.

COROLLARY 1.1. *Let $A = A_k \cdots A_{k+h(n)-1}$ and $B = B_{k'} \cdots B_{k'+h(n)-1}$ be two n-block $T_1(T_2)$ P-names, $n > 4$, $|k-k'| = h(n)-L$. Suppose $L > h(n)/2^{n-4}$, and $L \geqq K > h(n)/2^{n-4}$. If for some j, $\sup(k,k') \leqq j \leqq h(n) + \inf(k,k') - K$, and those $i \in \{j,\cdots,j+k\}$ we have $A_i \neq B_i$ on less than $\bar{\varepsilon}K/4$ values i, then $L > h(n) - \Sigma_{i=0}^{n}f(i)$.*

PROOF. Suppose there is such a K and j. The block of K places in A on which there are fewer than $\bar{\varepsilon}K/4$ errors must be at least half in complete $(n-1)$-blocks.

At least half of these must have fewer than $\bar{\varepsilon}h(n-1)$ errors. By Lemma 1.1, such an $(n-1)$-block must lie within $\Sigma_{i=0}^{n-1}f(i)$ of an $(n-1)$-block in B. This collection of such $(n-1)$-blocks must constain at least two blocks.

The distances between pairs of $(n-1)$-blocks in an n-block come in values always at least $s(n) > \Sigma_{i=0}^{n-1}f(i)$ apart. Hence these two blocks occupy the same position in B as they occupy in A. But then A and B must overlap in at least $h(n)-\Sigma_{i=0}^{n}f(i)$ places. When two n-block names $A_k \cdots A_{k+h(n)-1}$ and $B_{k'} \cdots B_{k'+h(n)-1}$ have the property that $|k-k'| < \Sigma_{i=0}^{n}f(i)$, we will say the blocks are *close*.

Thus if two blocks match well, even on a small segment of a block, then the blocks must be close. The next result tells us that if two n-blocks match well across some fraction of their overlap, not only are the blocks close, but along this fraction the match between 0-blocks is nearly perfect.We will again consider an overlap of size L, i.e. in the n-blocks $A = A_k \cdots A_{k+h(n)-1}$ and $B = B_{k'} \cdots B_{k'+h(n)-1}$, we will have $|k-k'| = h(n) - L$.

Let $(l,k)_A$ and $(l,k)_B$ be the kth l-blocks in A and B. Say (l,k) is close if the blocks $(l,k)_A$ and $(l,k)_B$ are close as l-blocks; that is to say, if A_i is the first point in the block $(l,k)_A$ and B_j is the first point in $(l,k)_B$, then $|j-i| < \Sigma_{i=0}^{l}f(i)$.

LEMMA 1.2. *Given* $\varepsilon > 0$, *if* A *and* B *are two* $T_1(T_2)$ *n-block names*, $n > 3 - ln(\varepsilon)$, $h(n) - |k'-k| = L'$, $h(n) \geqq L \geqq h(n) - \Sigma_{i=0}^{n}f(i)$, $L \geqq K \geqq h(n)/2^{n-4}$, *and* $A_i \neq B_i$ *for at most* $\varepsilon\bar{\varepsilon}K$ *values* $i \in (j, \cdots, j+k)$, *where* j *is some value* $\sup(k,k') \leqq j \leqq h(n) + \inf(k,k') - K$, *then at least* $(1-5\varepsilon)$ *of the complete* $(0,k)_A$ *in* $A_j \cdots A_{j+k}$ *are close to* $(0,k)_B$.

PROOF. By Corollary 1.1, $(n,1)$ is close. Let $\mathcal{L} = \{(l,k) \mid (l,k)$ is not close, but the $(l+1,k')$ containing (l,k) is$\}$. As $(n,1)$ is close, any (l,k) which is not close is contained in some $(l',k') \in \mathcal{L}$. Further, if (l,k) is close, then the $(l+1,k')$ containing (l,k) is also. Hence $\mathcal{L}$ is precisely the maximal non-closed blocks, and any $(0,k)$ block in an $(l,k') \in \mathcal{L}$ is not close.

For $(l,k) \in \mathcal{L}$, as the $(l+1,k')$ containing (l,k) is close, $(l,k)_A$ and $(l,k)_B$ must overlap in at least $h(l) - \Sigma_{i=0}^{l+1}f(i) > h(l)/2$ places. By Lemma 1.1 there are at least $\frac{1}{2}\bar{\varepsilon}h(l)$ errors between the A and B names across $(l,k)_A$. Let $\mathcal{L}' = \{(l,k) \in \mathcal{L}$ and $(l,k)_A$ lies in positions $j, \cdots, j+K$ in $A\}$.
As the (l,k) in $\mathcal{L}'$ are disjoint, we get at least $\frac{1}{2}\Sigma_{(l,k)\in\mathcal{L}'}h(l)$ errors between A and B across $j, \cdots, j+K$. Now noting that

$$\sum_{(l,k)\in\mathcal{L}'} h(l) \geqq h(0) \times (\text{number of non-close } (0,k) \text{ in } j, \cdots, j+k) - h(n-1),$$

we get

$$\varepsilon\bar{\varepsilon}K \geqq (h(0)/2) \times (\text{number of non-close } (0,k) \text{ in } j,\cdots,j+k) - h(n-1).$$

The number of 0-blocks in $j,\cdots,j+k$ is at least $K/2h(0)$. Hence the fraction of unaligned $(0,k)$ in $j,\cdots,j+k$ is at most $4\varepsilon + (4Kh(0)/h(n-1)) \leqq 4\varepsilon + 4/2^{n-1} < 5\varepsilon$ by our choice of n and the result follows.

LEMMA 1.3. *There is an $\varepsilon' > 0$ such that the following holds. For $n \geqq 8$, let A and B be $T_1(T_2)$ n-block names. Let $L > h(n)/2^{n-4}$ and $L \geqq K \geqq h(n)/2^{n-4}$. If for some j, $\sup(k,k') < j \leqq h(n) + \inf(k,k') - K$, and all $i \in \{j,\cdots,j+K\}$ we have $A_i \neq B_i$ on at most $\varepsilon' K$ values i, then $L > h(n) - \Sigma_{i=0}^{n} f(i)$, and A and B are both red or both black n-block names.*

PROOF. If $\varepsilon' < \bar{\varepsilon}$ and $n \geqq 5$, the first half of the result follows from Corollary 1.1. To get A and B with the same color let B' be an n-block name that has the same $(\bar{E}, \bar{F}, \bar{S}_0, \bar{S})$ name and indices as B, and the same color sequence as A. By Lemma 1.1, if $\varepsilon' < \varepsilon/20$ as $n \geqq 8$, at least 3/4 of the 0-blocks in A from j to $j+K$ are close to the corresponding ones in B', as B and B' have the same uncolored names. The uncolored names of A and B' differ in at most $\varepsilon' K$ places across j to $j+K$. Whenever a 0-block in A and B' in the same position are close, they have the same color. Hence A and B' have different colored names in at most $\frac{1}{4}K$ places across j to $j+K$. Thus B and B' have different colored names in at most $(\frac{1}{4} + 2\varepsilon')K$ places. Make sure $2\varepsilon' < 1/8$. As B and B' have the same uncolored names, B and B' have different colors only when they have different colors in every colored space. Hence our choice implies B and B' have the same color. Hence A and B do. This completes the lemma.

Now suppose there is a measurable, measure preserving ϕ with $\phi T_1 \phi^{-1} = T_2$. As ϕ is measurable, $\phi^{-1}(P) \subset \vee_{-\infty}^{\infty} T_1^i(P)$. Thus given any ε there is an $N(\varepsilon)$ with $\phi^{-1}(P) \subset^{\varepsilon} \vee_{-N(\varepsilon)}^{N(\varepsilon)} T_1^i(P)$ That is to say, there is a $\bar{P}(\varepsilon) \subset \vee_{-N(\varepsilon)}^{N(\varepsilon)} T_1^i(P)$, and $|\bar{P}(\varepsilon), \phi^{-1}(P)| < \varepsilon$. Hence, for almost every $w \in \bar{\Omega}$, the P-name of w from $-N(\varepsilon)+k$ to $N(\varepsilon)+k$ will determine what atom of $\bar{P}(\varepsilon)$, $T_1^k(w)$ is in, and this $\bar{P}(\varepsilon)$ name agrees with the P-name of $T_2^k(\phi(w))$ for all but a set of k's of density at most ε.

Let w be such a good point in $\bar{\Omega}$. As above, in the T_1, P-name of w, let A, B, C etc. denote occurrences of an n-block P-name in the P-block of w, i.e. a set of points $T_1^k(w)\cdots T_1^{k+h(n)}(w)$ whose P-name is an n-block name. Further, let $\langle A\rangle_n$ or $\langle A-B\rangle_n$ or $\langle A-B-C\rangle_n$ denote the collection of all occurrences of the P-name of A in the P-name of w, or of $A-B$, or the triple $A-B-C$, without specifying the spacers between the blocks A and B and C, but with A and B

and C all in the same $(n+1)$-block. A block A will $\bar{P}(\varepsilon)$ code α-well if the $\bar{P}(\varepsilon)$ name of $T_1^k(w)\cdots T_1^{k+h(n)}(w)$, and the P-name of $T_2^k(\phi(w))\cdots T_2^{k+h(n)}(\phi(w))$ differ in at most $\alpha h(n)$ places. The following lemmas now show that such a ϕ must preserve much of the block structure. Notice that Lemmas 1.4, 1.5 and 1.6 do not depend critically on the coloring, but rather on the block structure.

LEMMA 1.4. *For all but at most $\sqrt[3]{2\varepsilon}$ of the classes $\langle A\rangle_n$, for all $A\in\langle A\rangle_n$ but a set of density at most $\sqrt[3]{2\varepsilon}$, A $\bar{P}(\varepsilon)$ codes $\sqrt[3]{2\varepsilon}$-well. Similarly for the classes $\langle A-B\rangle_n$ and $\langle A-B-C\rangle_n$.*

PROOF. If not, then the density of errors is at least $(\mu(\Omega_n)/\mu(\Omega))\sqrt[3]{2\varepsilon}^3>\varepsilon$, a conflict.

LEMMA 1.5. *Given $\hat{\varepsilon}>0$, there is an N, such that for $n>N$, for all but at most $\hat{\varepsilon}$ of the classes $\langle A\rangle_n$, $\langle A-B\rangle_n$ and $\langle A-B-C\rangle_n$, for all but a set of n-blocks A in a class of density at most $\hat{\varepsilon}$, A maps by ϕ to the interior of an $(n+1)$-block.*

PROOF. This follows as $(h(n)/h(n+1))\to 0$ and $\mu(\Omega_n)\to\mu(\Omega)$.
We now begin to demonstrate the rigidity of the block structure under isomorphisms.

LEMMA 1.6. *Given any ε'' and i, if N is large enough, all but at most ε'' of the classes $\langle A\rangle_n$, for all but a set of $A\in\langle A\rangle_n$ of density ε'', $\phi(A)$ contains $(i-1)/i$ of an n-block in the T_2 name for $n>N$.*

PROOF. From Lemmas 1.4 and 1.5, for n large enough, all but $2\sqrt[3]{2\varepsilon}+\hat{\varepsilon}$ of the n-block pairs $\langle A-B\rangle_n$, $\phi(A-B)$ lie in the interior of an $(n+1)$-block with both A and B $\bar{P}(\varepsilon)$ coded $4\sqrt[3]{2\varepsilon}$-well for all but $\sqrt[3]{2\varepsilon}+\hat{\varepsilon}$ of the pairs $A-B$. The pair $A-B$ can occur with any even (or odd) spacing in $s(n+1)-1$, $2s(n+1)-1,\cdots,(2^{2(n+1)}-1)s(n+1)-1$, each with equal probability, which by ergodicity is equivalent to density in $\langle A-B\rangle_n$. Choose ε and $\hat{\varepsilon}$ so that for all n, $(\hat{\varepsilon}+4\sqrt[3]{2\varepsilon}<(2^{n+1}-2)/(2^{n+1}-1)$ and $(5/((1/2i)-(1/(2i)^8)))\sqrt[3]{4\varepsilon}<\varepsilon'$ of Corollary 1.1. Then there must be two occurences in $A-B$ which have both A and B $\bar{P}(\varepsilon)$ coded $4\sqrt[3]{2\varepsilon}$ well to the interior of an $(n+1)$-block, but where the spacings between A and B differ. Call these occurences $A-B$ and $A'-B'$ with spacings $ks(n+1)$ and $k's(n+1)$ respectively. Under these circumstances, suppose $\phi(A)$ contains less than $((2i-1)/2i)h(n)$ of the n-block C it overlaps, as it codes into an $(n+1)$-block (see Fig. 6).

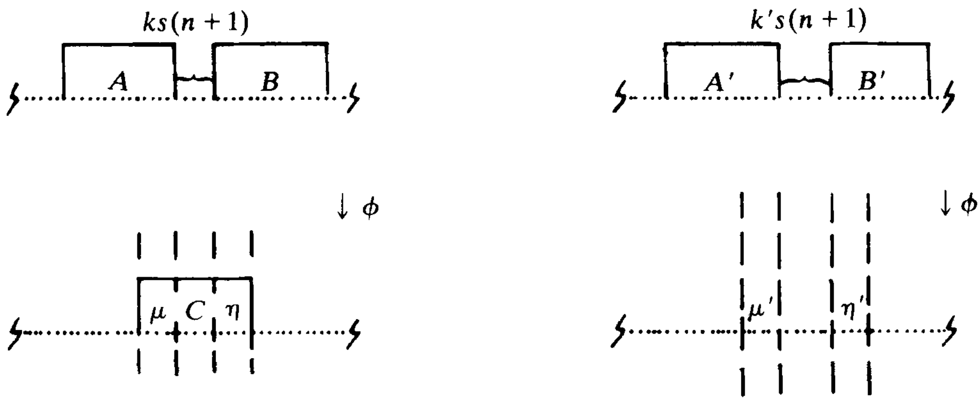

Fig. 6

If n is large enough that $2i < 2^{n/10}$, then $\phi(B)$ contains at least

$$\frac{h(n)}{2i} - \left(\frac{h(n)}{(2i)^8}\right) > \frac{h(n)}{2^n}$$

of the block C. Let μ and η be the two sections of C that A and B overlap, and μ' and η' the corresponding sections in $\phi(A')$ and $\phi(B')$. As A, B, A' and B' all code $4\sqrt[3]{4\varepsilon}$-well μ and μ' differ in at most

$$2N(\varepsilon') + 4\sqrt[3]{4\varepsilon}h(n) < \varepsilon'\left(\frac{h(n)}{2i + (2i)^8}\right)$$

if n is large enough that $N(\varepsilon) < \sqrt[3]{\varepsilon}h(n)$. If n is large enough, as they lie inside an $(n+1)$-block, μ' and η' both intersect n-blocks in at least $h(n)/2^{n+1}$ places. The above inequality says μ' and η' must, by Corollary 1.1, lie in these n-blocks within $\Sigma_{i=0}^{n} f(i)$ of the way they lie in C. But as $h(n-1)/2^{10n} > s(n) > \Sigma_{i=0}^{n} f(i)$, no such blocks could exist. Hence $\phi(A)$ and likewise $\phi(B)$ must contain at least $((2i-1)/2i)h(n)$ of an n-block. Hence, for n large enough of all the $\langle A \rangle_n$, at least $1 - (2\sqrt[3]{2\varepsilon} + \hat{\varepsilon})$ contain an A, coded $\sqrt[3]{2\varepsilon}$ well for which $\phi(A)$ overlaps an n-block in at least $((2i-1)/2i)h(n)$ places. But if any other $A' \in \langle A \rangle$ codes $\sqrt[3]{2\varepsilon}$ well into the interior of an $(n+1)$-block, then as above it must also overlap an n-block within $\Sigma_{i=0}^{n} f(i)$ of the way A does, hence overlap in at least

$$((2i-1)/2i)h(n) - \sum_{i=0}^{n} f(i) > ((i-1)/i)h(n)$$

places. Thus with $\sqrt[3]{2\varepsilon} + \hat{\varepsilon} < \varepsilon''$ we get the result.

In fact, if C and C' are the n-blocks that A and A' overlap, and $A = T_1^k(A')$, then $T_2^{-k}(C)$ and C' agree in at least $1 - 2\sqrt[3]{2\varepsilon} - (1/i)$ places. Such A make up all but $2\sqrt[3]{2\varepsilon} + \hat{\varepsilon}$ of the classes. Hence, we also get the following fact.

LEMMA 1.7. *If n is large enough, and A and $A' = T_1^k(A) \in \langle A \rangle_n$, both of which $\bar{P}(\varepsilon)$ code $\sqrt[3]{2\varepsilon}$ well and $\phi(A)$ and $\phi(A')$ overlap n-blocks C and C' respectively, each in more than $(3/4)h(n)$ places, then $T_2^k(C)$ overlaps C' in at least $(1 - \Sigma_{i=0}^n f(i))h(n)$ places, and C and C' have the same color.*

PROOF. This follows from the above comment and Lemma 1.2.

We will say a class $\langle A \rangle_n$ is coded α-very well if for all but α of the $A \in \langle A \rangle$, $\phi\langle A \rangle$ overlaps an n-block C in $(3/4)h(n)$ places, and any two such overlapped blocks C and C' are overlapped in sections differing by at most $\Sigma_{i=0}^n f(i)$, and have the same color.

COROLLARY 1.2. *Given $\alpha > 0$, if n is large enough, all but α of the n-block classes $\langle A \rangle_n$ code α-very well.*

THEOREM 1.3. *The maps T_1 and T_2 are nonisomorphic.*

PROOF. We argue by contradiction. Suppose an isomorphism ϕ exists. Define the following map Φ on classes of n-block T_1, P-names in the T_1, P-name of w. In an n-block P-name take the $(n-1)$-block names in positions $4k+2$ and $4k+3$ and switch them, and fix the names in positions $4k+1$ and $4k+4$. Let $\Phi(\langle A \rangle)$ be the class of occurrences of the name formed from A by this process. This map takes classes of n-block T_1, P-names to each other (but notice not T_2, P-names).

In an n-block name define a quadruple of $(n-1)$-block names as those occupying positions $4k+1$, $4k+2$, $4k+3$ and $4k+4$. Let $\alpha = 1/2^{12}$ and choose N by Corollary 1.2 so that (*) the density in the T, P-name of w of n-block quadruples, all four of which code $1/2^{12}$-well into the same $(n+1)$-block is at least

$$\underbrace{\mu(\Omega_n)\left(1-\frac{8}{2^{12}}\right)}_{\substack{\text{density of quadruples}\\ \text{all coded } 1/2^{12} \text{ very well}}} \quad - \quad \underbrace{\mu(\Omega_n)\left(\frac{2}{2^{12}}\right)}_{\substack{\text{density of } (n+1)\\ \text{blocks not coding}\\ \text{to an overlap of}\\ \text{length at least}\\ (3/4)h(n+1)}} \quad - \quad \underbrace{\left(\frac{1}{4}-\frac{1}{4^{n-2}}\right)}_{\substack{\text{density of those}\\ \text{quadruples in } (n+1)\\ \text{blocks which overlap}\\ \text{in } (3/4)h(n+1) \text{ places,}\\ \text{but not in this overlap}}}$$

$\geqq 5/8$ for $n \geqq N$.

Now consider A, B, C, D, a quadruple in $\bar{A} \in \langle \bar{A} \rangle$, and A', B', C', D', the image quadruple in some $\bar{A}' \in \Phi(\langle \bar{A} \rangle)$. Suppose $A, B, C, D, A', B', C', D'$, all code $1/2^{12}$-well and into the same $(n+1)$-blocks respectively (see Fig. 7).

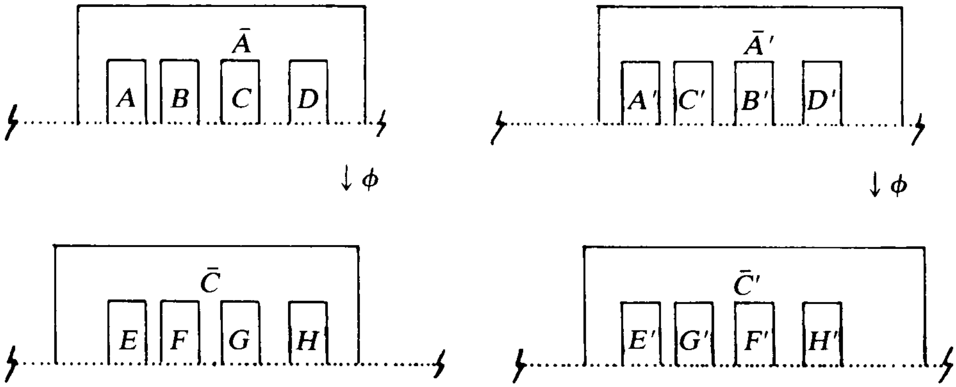

Fig. 7

Let $E, F, G, H, E', F', G', H'$, be the n-blocks indicated in the diagram in the image of $\bar{A}$ and $\bar{A}'$. E lies below A within $\Sigma_{i=0}^{n} f(i)$ of how E' lies below A', similarly F below B and F' below B', G below C and G' below C', H below D and H' below D'. As $s(n+1) > \Sigma_{i=0}^{n} f(i)$, this forces E to lie in $\bar{C}$ in the same n-block position as A in $\bar{A}$, similarly E' lies in $\bar{C}'$ as the same n-block as A' in $\bar{A}'$. Then E, F, G, H must have colors rrbb or bbrr. Further, E', F', G', H', have the same colors as E, F, G, H. But then in $\bar{C}'$ the colors have order rbrb or brbr, neither of which ever occur. Thus both quadruples A, B, C, D and A', B', C', D' could not be coded $1/2^{12}$-very well into the same $(n+1)$-blocks. Thus the density of such quadruples is at most 1/2, conflicting with (*). Hence no isomorphism could exist.

This example also provides a counterexample to a conjecture of K. Berg [1]. Suppose we can write a transformation in two ways as 0-entropy $\times K$, i.e. as $T \times K_1$ and $T \times K_2$ where T is the zero entropy part and K_1 and K_2 are K-automorphisms. Is K_1 then isomorphic to K_2? The answer is yes if either K_1 or K_2 is known to be Bernoulli, from the work of J. P. Thouvenot [10]. But let T be the switch map on a space $\{a, b\}$, and let $K_1 = T_1$ and $K_2 = T_2$. Define

$$\phi : (a, b) \times \Omega \times (0,1) \rightarrow (a, b) \times \Omega \times (0,1) \text{ by } \begin{cases} \phi(\{a, w, 0\}) = \{a, w, 0\} \\ \phi(\{a, w, 1\}) = \{a, w, 1\} \\ \phi(\{b, w, 0\}) = \{b, w, 1\} \\ \phi(\{b, w, 1\}) = \{a, w, 0\}. \end{cases}$$

Then by either looking at how ϕ affects names or computing it out, $\phi(T \times T_1) = T \times T_2(\phi)$, and ϕ is the identity map on the zero entropy factor $\{a, b\}$. Thus the transformation $T \times T_1$ can be written in two ways as zero entropy cross a K-automorphism but the two K factors are not isomorphic. The Pinsker algebra in this case is terribly simple. It would be interesting to try to get such an example where the Pinsker algebra is mixing.

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